Hence, f(H) is a normal subgroup of G′.

3.1.8. Kernel of Homomorphism

Definition: If f is a homomorphism of a group G into a group G′, then the set K of all those elements of G which are mapped by f onto the identity e′ of G′ is called the kernel of the homomorphism f.

Let us consider (G, o) and (G', *) are two groups and $f : G \rightarrow G'$ is a homomorphism. Thus the kermel of f is a subset of G. The kernel of f is denoted by ker f.

It is defined by, $\ker f = \{a \in G : f(a) = e'\}$.

Thus ker f is the set of those elements of G that are mapped to the identity element of G' under the homomorphism f.

For examples,

1) Let us consider the cyclic group Z/3Z = {0, 1, 2} and the group of integers Z with addition. The map h: Z → Z/3Z with h(u) = u mod 3 is a group homomorphism. It is surjective and its kernel involves all integers that are divisible by 3.

2) In addition to the group of non-zero complex numbers C* with exponential operation multiplication, the group of complex numbers C also produces group homomorphism.

3) If two groups G and H are given then the map h: G → H that sends each element of G to the identity element of H is a homomorphism. Its kernel is all of G.

Theorem of Kernel of Homomorphism

Theorem (First Homomorphism Theorem) 5: Let $f: G \to H$ be a group homomorphism, prove that Ker(f) is a normal subgroup of G.

Proof: According to definition we have,

$\text{Ker } f = \{x \in G : f(x) = e',\} = K$, where $e' \in H$

There is a need to prove that Ker f is a subgroup of G.

Suppose $x, y \in \text{Ker } f \Rightarrow f(x) = e', f(y) = e'$

Consider $f(xy^{-1}) = f(x)\, f(y^{-1})$ [homomorphism]

$= f(x)\,(f(y))^{-1} = e'(e')^{-1} = e'$

$\Rightarrow xy^{-1} \in \text{Ker } f \Rightarrow$ Ker f is a subgroup of G.

Suppose $g \in G$ and $x \in \text{Ker } f$,

Consider $f(gxg^{-1}) = f(g)\, f(xg^{-1})$ [f is homomorphism]

$= f(g)\, f(x)\, f(g^{-1}) = f(g)\, f(x)\, (f(g))^{-1}$

$= f(g)\, e'(f(g))^{-1} = f(g)\, .\, (f(g))^{-1} = e'$

$\Rightarrow gxg^{-1} \in \text{Ker } f \Rightarrow$ Ker f is a normal subgroup of G.

Theorem 6: Let (G, o) and (G', *) be two groups and $f: G \to G'$ is a homomorphism. Then f is one-to-one if and only if ker f = {e}, e being the identity element of G.

Proof: Suppose f is an one-to-one then take an arbitrary element $a \in \ker f$. If e' is the identity element of G', then $f(a) = e' = f(e)$ [$\therefore f(e) = e'$] $\Rightarrow a = e$ [$\therefore$ f is one-one]

Subsequently a is an arbitrary element of ker f and $a \in \ker f \in a = e$, satisfies $\ker f = \{e\}$.

On the other hand, suppose $\ker f = \{e\}$. Let us take two arbitrary elements $a, b \in G$.

As f is homomorphism, hence,

$f(a) = f(b) \Rightarrow f(a) * (f(b))^{-1} = f(b) * (f(b))^{-1} \Rightarrow f(a) * f(b^{-1}) = e'$

$\Rightarrow f(a \text{ o } b^{-1}) = e' \Rightarrow a \text{ o } b^{-1} \in \ker f \Rightarrow a \text{ o } b^{-1} = e \Rightarrow a = b.$

Therefore f is one-to-one and $a^{-1} \in$ ker (g) and ker (g) is a sub-group of (G,*).

The